Multiple Parameter Structure of Mielnik's Isospectrality in Unbroken SUSYQM

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Within unbroken SUSYQM and for zero factorization energy, I present an iterative generalization of Mielnik's isospectral method by employing a Schrödinger true zero mode in the first-step general Riccati solution and imposing the physical condition of normalization at each iterative step. This procedure leads to a well-defined multiple-parameter structure within Mielnik's construction for both zero modes and potentials.

The supersymmetric procedures are an interesting and fruitful extension of (one-dimensional) quantum mechanics. For recent reviews see ref. 1. These techniques are, essentially, factorizations of one-dimensional Schrödinger operators, first discussed in the supersymmetric context by Witten in 1981 [2], and well known in the mathematical literature in the broader sense of Darboux covariance of Schrödinger equations [3].

In 1984, Mielnik [4] introduced a different factorization of the quantum harmonic oscillator based on the general Riccati solution. As a result, Mielnik obtained a one-parameter family of potentials with *exactly* the same spectrum as that of the harmonic oscillator. However, even though in the same year Nieto discussed the connection of such a factorization with the inverse scattering approach, and Fernández applied it to the hydrogen atom case, Mielnik's result remained a curiosity for a decade during which only a few authors paid attention to it. On the other hand, constructing families of strictly isospectral potentials is an important possibility with many potential applications in physics [1]. This explains the recent surge of interest in this supersymmetric issue [5]. My goal in this work is to give a multiple-parameter generalization of Mielnik's procedure based on the ground-state function of any soluble

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one-dimensional quantum mechanical problem. This is just a form of Crum's iterations, i.e., repeated Darboux transformations. Some work along this line has already been done by Keung *et al.* [6], who performed an iterative construction for the reflectionless, solitonic, sech potentials and the attractive Coulomb potential, presenting relevant plots as well. However, they first go *n* steps away from a given ground state and only afterward perform the *n* steps backward. On the other hand, Pappademos *et al.* [7], working in the continuum part of the spectrum, got one- and two-parameter supersymmetric families of potentials strictly isospectral with respect to the half-line free particle and Coulomb potentials and focused on the supersymmetric bound states in the continuum. Their procedure is closer to the method I will present in the following. For more recent work see Bagrov and Samsonov [8], Fernández *et al.* [9], Junker and Roy [10], and Rosas-Ortiz [11].

In the following, I first briefly recall the mathematical background of Mielnik's method and next pass to a simple multiple-parameter generalization for the particular but physically relevant zero-mode case.

I begin with the "fermionic" Riccati (FR) equation $y' = -y^2 + V_1(x)$ [the "bosonic" one being $y' = y^2 + V_0(x)$] for which I suppose to know a particular solution y_0. Notice also that I do not put any free constant in the Riccati equations, that is, I work at zero factorization energy. Let us seek the general solution in the form $y_1 = w_1 + y_0$. By substituting y_1 in the FR equation, one gets the Bernoulli equation $-w_1' = w_1^2 + (2y_0)w_1$. Furthermore, using $w_2 = 1/w_1$, we obtain the simple first-order linear differential equation $w_2' - (2y_0)w_2 - 1 = 0$, which can be solved by employing the integration factor $F_0(x) = \exp(-\int_c^x 2y_0)$, leading to the solution $w_2(x) = (\lambda + \int_c^x F_0(z) \, dz)/F_0(x)$, where λ occurs as an integration constant. In applications the lower limit c is either $-\infty$ or 0 depending on whether one deals with full-line or half-line problems, respectively. In the latter case, λ is restricted to be a positive number. Thus, the general FR solution reads

$$y_1 = y_0 + \frac{\exp(-\int_c^x 2y_0)}{\lambda + \int_c^x \exp(-\int_c^x 2y_0)}$$

$$\equiv y_0 + \frac{F_0}{\lambda + \int_c^x (F_0)}$$

$$= y_0 + D \ln\left(\lambda + \int_c^x (F_0)\right) \tag{1}$$

where $D = d/dx$. It is easy to reach the conclusion that the particular FR solution y_0 corresponds to Witten's superpotential [2], while the general FR solution $y_1 = y_0' + F_0/(\lambda + \int_c^x F_0)$ is of Mielnik type [4]. This is especially clear when one is able to identify $f_0 = F_0^{1/2}$ with the quantum mechanical ground-state wavefunction u_0 of the problem at hand. This requires suitable asymptotic behavior of the Riccati solution y_0 and applying the normalization condition to f_0, turning it into a true zero mode. As is well known, this case corresponds to the so-called unbroken SUSYQM, which will be assumed to hold henceforth. Moreover, $-2y_0'$ $(\equiv -2(d^2/dx^2) \ln F_0^{1/2})$ is the Darboux transform contribution to the initial Schrödinger potential, i.e., $V_1 = V_0 - 2y_0'$. Also, the modes

$$u_\lambda(x) = \frac{F_0^{1/2}}{\lambda + \displaystyle\int_c^x F_0} = \frac{u_0}{\lambda + \displaystyle\int_c^x u_0^2} \tag{2}$$

can be normalized and therefore considered as ground-state wavefunctions of the bosonic family of potentials corresponding to Mielnik's parametric superpotential. The one-parameter true zero modes read

$$v_\lambda(x) = \frac{\sqrt{\Lambda}F_0^{1/2}}{\lambda + \displaystyle\int_c^x F_0} = \frac{\sqrt{\Lambda}u_0}{\lambda + \displaystyle\int_c^x u_0^2} \tag{3}$$

where $\sqrt{\Lambda} = \sqrt{\lambda(\lambda + 1)}$ is the normalization constant.

Moreover, $-2y_1'$ can be thought of as the general Darboux transform contribution to the initial potential generating the bosonic strictly isospectral family, which reads

$$V_\lambda^M = V_0(x) - 2\frac{d^2}{dx^2} \ln\left(\lambda + \int^x u_0^2\right) = V_0(x) - \frac{4u_0u_0'}{\lambda + \displaystyle\int^x u_0^2} + \frac{2u_0^4}{\left(\lambda + \displaystyle\int^x u_0^2\right)^2} \tag{4}$$

This family of potentials can be seen as a continuous deformation of the original potential, because the latter is included in the infinite limit of the deforming parameter λ and $v_{\pm\infty} = u_0$ as well. In more intuitive terms, Mielnik's method based on an initial Schrödinger true zero mode may be called a double Darboux technique of deleting followed by reinstating a nodeless ground-state wavefunction $u_0(x)$ of a potential $V_0(x)$ by means of which one can generate a one-parameter family of isospectral potentials $V_\lambda(x)$, where λ is a labeling, real parameter of each member potential in the set.

One can go on with one of the strictly isospectral bosonic zero modes $u_{\lambda_1} = u_0/(\lambda_1 + \int_c^x u_0^2)$ (i.e., by choosing $\lambda = \lambda_1$) and repeat the strictly isospectral construction, getting a new two-parameter zero mode

$$u_{\lambda_1,\lambda_2} = \frac{u_{\lambda_1}}{\lambda_2 + \int_c^x u_{\lambda_1}^2} = \frac{u_0}{\left(\lambda_1 + \int_c^x u_0^2\right)\left(\lambda_2 + \int_c^x u_{\lambda_1}^2\right)} \tag{5}$$

The two-parameter true zero modes read

$$v_{\lambda_1,\lambda_2} = \frac{\sqrt{\Lambda_1\Lambda_2}\, u_0}{\left(\lambda_1 + \int^x u_0^2\right)\left(\lambda_2 + \int^x v_{\lambda_1}^2\right)} \tag{6}$$

The resulting two-parameter family of strictly isospectral potentials will be

$$V_{\lambda_1,\lambda_2} = V_0 - 2\frac{d^2}{dx^2}\ln\left[\left(\lambda_1 + \int^x u_0^2\right)\left(\lambda_2 + \int^x v_{\lambda_1}^2\right)\right] \tag{7}$$

At the ith-parameter level, one will have

$$V_{\lambda_1,\lambda_2,\dots\lambda_i} = V_0 - 2\frac{d^2}{dx^2}\ln\left[\left(\lambda_1 + \int^x u_0^2\right)\left(\lambda_2 + \int^x v_{\lambda_1}^2\right)\right.$$
$$\left. \dots \left(\lambda_i + \int^x v_{\lambda_1\dots\lambda_{i-1}}^2\right)\right] \tag{8}$$

and

$$v_{\lambda_1\dots\lambda_i} = \frac{\sqrt{\Lambda_1\Lambda_2\dots\lambda_i}\, u_0}{\left(\lambda_1 + \int^x u_0^2\right)\dots\left(\lambda_i + \int^x v_{\lambda_1\dots\lambda_{i-1}}^2\right)} \tag{9}$$

Explicit formulas for the parametric zero modes can be obtained if one uses a notation based on the integration factor $\int_c^x F_0 = \mathscr{F}(x) - \mathscr{F}(c) = \Delta_x\mathscr{F}$. Then

$$v_{\lambda_1}(x) = \frac{\sqrt{\Lambda_1}\, u_0}{\lambda_1 + \Delta_x\mathscr{F}} \tag{10}$$

Next, one can calculate

$$\int_c^x \frac{u_0^2}{\left(\lambda + \int_c^x u_0^2\right)^2} = \int_c^x \frac{\mathscr{F}' \, dx}{(\lambda - \mathscr{F}(c) + \mathscr{F}(x))^2}$$

$$= \int_{\mathscr{F}(c)}^{\mathscr{F}(x)} \frac{dz}{(\lambda - \mathscr{F}(c) + z)^2}$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda + \Delta_x \mathscr{F}} = \frac{\Delta_x \mathscr{F}}{\lambda(\lambda + \Delta_x \mathscr{F})} \tag{11}$$

Thus

$$v_{\lambda_1, \lambda_2}(x) = \frac{\sqrt{\Lambda_1 \Lambda_2} u_0}{\lambda_1 \lambda_2 + (\lambda_1 + \lambda_2 + 1)\Delta_x \mathscr{F}} \tag{12}$$

At the next step one gets

$$v_{\lambda_1, \lambda_2, \lambda_3}(x)$$

$$= \frac{\sqrt{\Lambda_1 \Lambda_2 \Lambda_3} u_0}{\lambda_1 \lambda_2 \lambda_3 + (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 + \lambda_2 + \lambda_3 + 1)\Delta_x \mathscr{F}} \tag{13}$$

and the general formula at the i level can be written down in the form

$$v_{\lambda_1, \lambda_2, \ldots, \lambda_i}(x) = \frac{\sqrt{\Lambda_1, \ldots \Lambda_i} u_0}{C_1^{(i)} + C_2^{(i)} \Delta_x \mathscr{F}} \tag{14}$$

where the first coefficient in the denominator is the product of all parameters, whereas the second coefficient is just the sum over all the rest of lower order Viete-type expressions in the parameters. By the same token, one can write a general formula for the strictly isospectral potentials

$$V_{\lambda_1, \lambda_2, \ldots, \lambda_i} = V_0 - 2D^2 \ln[C_1^{(i)} + C_2^{(i)} \Delta_x \mathscr{F}]$$

$$= V_0 - \frac{4C_2^{(i)} u_0 u_0'}{C_1^{(i)} + C_2^{(i)} \Delta_x \mathscr{F}} + \frac{2(C_2^{(i)})^2 u_0^4}{(C_1^{(i)} + C_2^{(i)} \Delta_x \mathscr{F})^2} \tag{15}$$

which may be considered as the generalization of the furthest right-hand side of Eq. (4) and represents a simple generalization of Mielnik's one-parameter potentials.

Since $\Lambda_1 \ldots \Lambda_i = C_1^{(i)}(C_1^{(i)} + C_2^{(i)})$, one might think that there is nothing new in (14) and (15) with respect to a common Mielnik solution with an effective parameter $\lambda_{\text{eff}}^{(i)} = C_1^{(i)}/C_2^{(i)}$. However, I will argue that by performing such an equivalence one loses a certain type of information. This information is a consequence of the symmetry of (14) and (15) in the space of parameters. One can see that the subindices of any pair of parameters can be interchanged

without affecting the formulas. Thus, each λ parameter can be varied independently of the others, making it possible to put questions related to the following type of situation. Suppose we construct two Mielnik potentials corresponding to λ_1 and λ_2 and ask what is the potential bearing true zero modes that for $\lambda_1 \to \pm\infty$ goes to the Mielnik case for λ_2, whereas for $\lambda_2 \to \pm\infty$ it goes to the Mielnik case for λ_1. The answer is provided by the construction of this work and corresponds to the particular case V_{λ_1,λ_2} bearing the true zero modes v_{λ_1,λ_2}. Indeed, as one can easily check, $v_{\lambda_1,\pm\infty} = v_{\lambda_1}$ and $V_{\lambda_1,\pm\infty} = V_{\lambda_1}^M$, whereas $v_{\pm\infty,\lambda_2} = v_{\lambda_2}$ and $V_{\pm\infty,\lambda1} = V_{\lambda_2}^M$. In the general case, one starts with a set of i Mielnik potentials corresponding to i fixed values of Mielnik's parameter and asks the same question, this time for the set of i asymptotic limits. The answer is given by (14) and (15) and cannot be provided if one works with only one effective parameter unless its multiple-parameter value found above is used.

Another interesting remark is the one-to-one relationship between any polynomial equation $a_0 x^i + a_1 x^{i-1} + \ldots + a_i = 0$ and the present iterative construction. If we consider the λ parameters as the zeros of such arbitrary polynomials, we can write (14) as

$$v_{\lambda_1,\lambda_2,\ldots,\lambda_i} = \frac{\sqrt{(-1)^i a_i (\Sigma_0^i\,(-1)^i a_i)} u_0}{(-1)^i a_i + (\Sigma_0^{i-1}\,(-1)^i a_i)\Delta_x \mathcal{F}} \tag{16}$$

and in (15) one can substitute the same type of denominator as in (16). There is only one constraint on the employed polynomials, which one should impose in order to avoid possible singularities. Usually the integral $\Delta_x \mathcal{F}$ in the denominators of (15) and (16) is of the kink type, i.e., it may be written in the form $\alpha + \beta K(x)$, where the function $K(x)$ has the kink behavior, taking values between -1 and $+1$, and α and β are some constants, of which α may be zero. Then the allowed intervals for the effective parameter are $\lambda_{\mathrm{eff}}^{(i)} > \beta - \alpha$ and $\lambda_{\mathrm{eff}}^{(i)} < -(\beta + \alpha)$. When $\alpha = 0$ one gets $|\lambda_{\mathrm{eff}}^{(i)}| > \beta$.

It is worthwhile to mention that the previous iteration process can be understood most easily from the Riccati equation standpoint as follows. To get, for example, the two-parameter zero mode, one should start again with the FR equation $y' = -y^2 + V_1(x)$ and take as the known particular solution $y_p^{(1)} = y_0 + y_{\lambda_1}$, where $y_{\lambda_1} = F_0/[\lambda_1 + \int_c^x (F_0)]$. The intermediate Bernoulli equation will be $-w_1' = w_1^2 + 2y_p^{(1)}w_1$. This is turned into a first-order differential equation by the inverse function method. The integration factor of the latter is $F_{\lambda_1} = \exp(-\int_c^x 2y_p^{(1)})$ and the solution for the first-order differential equation can be written $w_2 = (\lambda_2 + \int_c^x F_{\lambda_1}dz)/F_{\lambda_1}$. From this presentation it is clear how one should proceed for an arbitrary step. Also, the logarithmic derivative notation in Eq. (1) is equally convenient to have a clear image
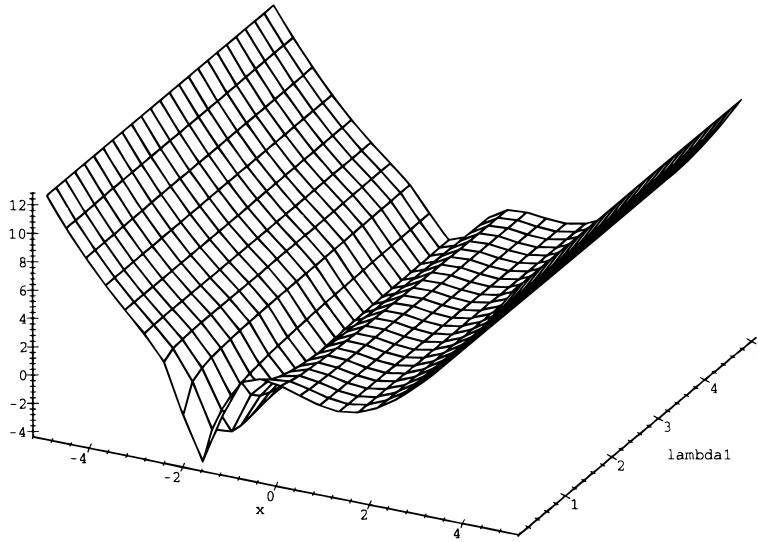
Fig. 1. Two-parameter strictly isospectral harmonic oscillator potentials ($m = \omega = 1$) for fixed $\lambda_2 = 0.2$ and $\lambda_1 \in [0.1, 5]$. They are identical to Mielnik harmonic oscillator potentials with $\lambda_{\text{eff}}^{(2)} \in [0.0154, 0.1613]$.

of the iteration process. Thus, one can generate hierarchies of parametric Schrödinger zero modes of any desired order by means of the Riccati connection.

The parametric normalization deletes the interval $[-1, 0]$ from the parameter space of $\lambda_{\text{eff}}^{(i)}$. At the -1 limit, one can make a connection with the Abraham–Moses isospectral technique [12], whereas at the 0 limit the connection can be done with another isospectral construction developed by Pursey [13]. This connection is only from the point of view of the potentials; the zero modes as worked out here just disappear.

In conclusion, I have shown explicitly the way Crum's iteration works when the general Riccati solutions (general superpotentials) at zero factorization energy are based on the corresponding Schrödinger ground-state wavefunctions, obtaining general formulas for this simple 'generalization' of Mielnik's one-parameter SUSYQM isospectrality. Plots of the two-parameter formulas for the harmonic oscillator case are presented in Fig. 1–5. One may consider the results of this work as pointing to an interesting hierarchical structure within the general Riccati solution produced by a particular type of repeated Darboux transformation when the normalization condition of quantum mechanics is taken care of at each iterative step.
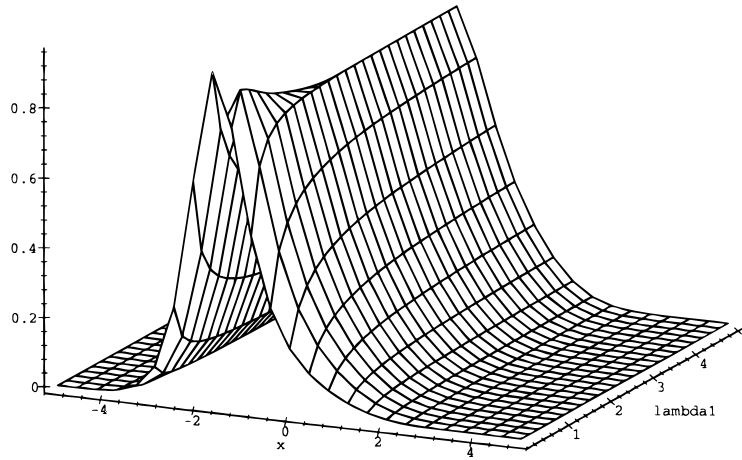
Fig. 2. The corresponding true zero modes.
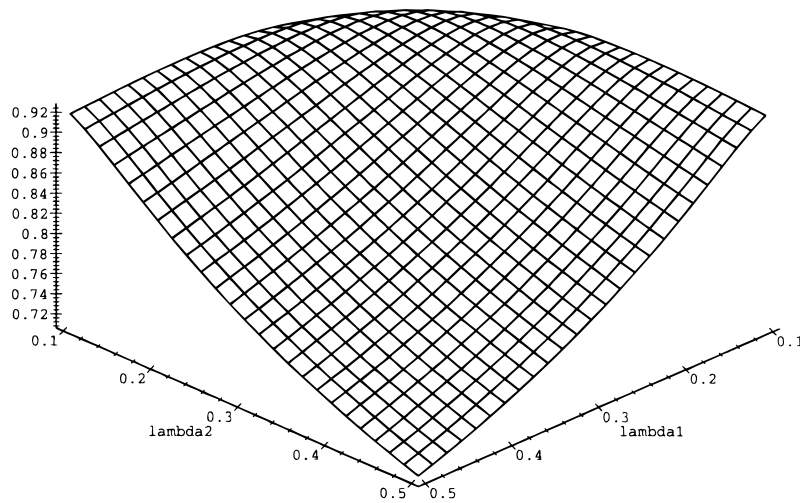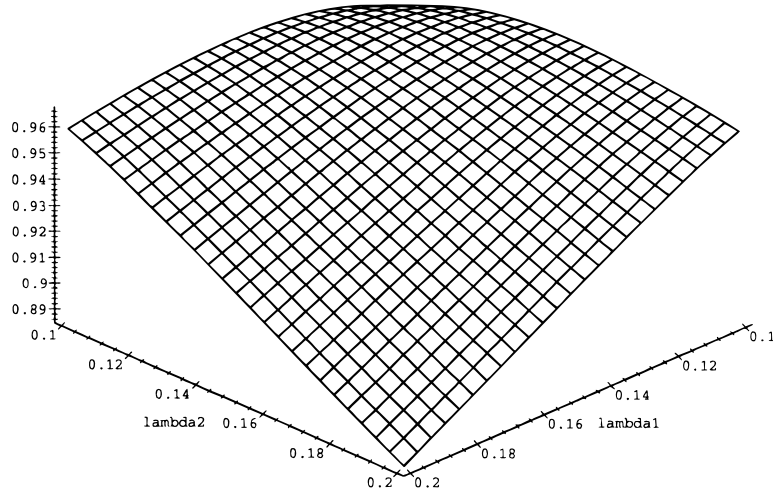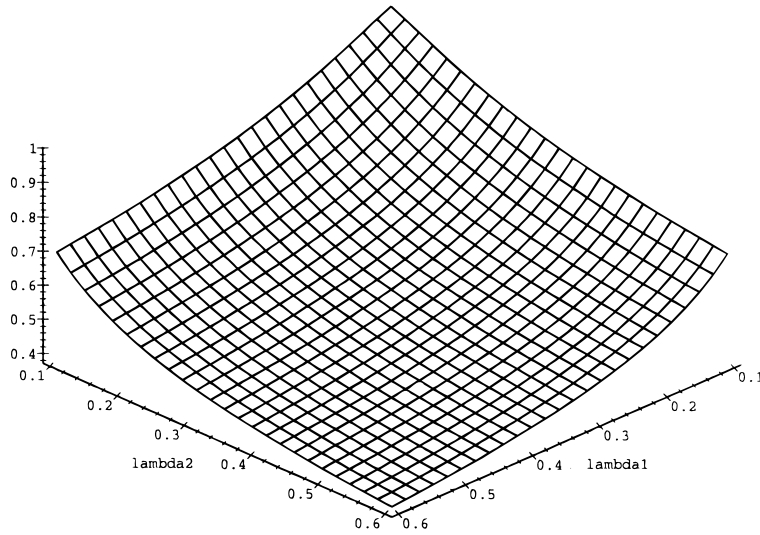
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Fig. 3. The two-parameter true zero modes at fixed $x = -1.4$ for the two-parameter formulas.

Fig. 4. Same modes as in Fig. 3 at fixed $x = -1.6$.

Fig. 5. Same modes as in Fig. 3 for fixed $x = -1.8$.

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